On Hypercomplex Number Systems in Seven Units.

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§1.

In a paper which recently appeared in the Mathematische Annalen I have given a method for enumerating all distinct classes of non-quaternion number systems in n units, at least two of which are idempotent, from systems in $n-1$ units. This paper supplements those of Starkweather* so that the enumeration problem for non-quaternion systems has now reached a complete solution. I hope in a future paper to complete the enumeration problem by giving a general enumeration of quaternion systems and to apply the complete result to various related subjects. In order to place in available form a source from which examples and illustrations may be drawn, and to place on record the actual enumeration of systems of as high an order as seems at present desirable, I have in the present paper given the enumeration of distinct types of irreducible, non-reciprocal, non-quaternion systems with moduli in seven units at least two of which are idempotent.† In §2 a general device is obtained which removes the necessity for considerable ineffective labor in making the enumeration which my outline in the Annalen required.

§2.

Method of Obtaining Tables of Combination.

An outline of the method to be followed in the enumeration is indicated by the following theorems.‡

* American Journal of Mathematics, Vols. 21 and 23.

† Systems in less than five units are enumerated by Scheffers, Mathematische Annalen, Vol. 39. Systems in six units, in one idempotent unit, are enumerated by Starkweather, American Journal of Mathematics, Vol. 23. Those in more than one idempotent unit I have given, loc. cit.

‡ Proofs of these theorems may be found in my papers in Transactions of the American Mathematical Society, Vol. 3, and Mathematische Annalen, loc. cit.

1. *Every unit in a non-quaternion system falls into one of the four following groups with respect to each of the idempotent units e_k of the system:*

Group I_k contains those units e_i such that

$$e_i e_k = e_k e_i = e_i$$

Group II_k contains those units e_i such that

$$e_i e_k = 0, \quad e_k e_i = e_i$$

Group III_k contains those units e_i such that

$$e_i e_k = e_i, \quad e_k e_i = 0$$

Group IV_k contains those units e_i such that

$$e_i e_k = e_k e_i = 0$$

2. *If two non-quaternion systems do not contain the same number of idempotent units, they are inequivalent.*

Thus we may enumerate all systems with a given number of idempotent units without including those enumerated with a different number of such units.

3. *If S and S' are equivalent there is a one to one correspondance between the idempotent units e_{r+i} and e'_{r+i} $(i = 1, 2, \ldots, n-r)$, such that the number of units in the groups I_k, II_k, III_k, IV_k, and I'_k, II'_k, III'_k, IV'_k are respectively the same where e_k and e'_k are corresponding units.*

This theorem shows that the first step toward enumeration is the formation of a table which gives, for a given value of r the different combinations of non-idempotent units into groups. We erase from this table all combinations that would lead to reducible systems and one of each pair of combinations that would afford reciprocal systems in accordance with the following principles.

4. *The necessary and sufficient condition that a system is reducible is that its modulus falls into parts each of which is the modulus of a certain subsystem.*

5. *Those combinations that are identical except that the number of units in groups II and III with respect to every idempotent unit are mutually interchanged, lead to reciprocal systems.*

Thus if in a given system there are in groups II_k and III_k, λ and λ' units respectively there will be in groups II_k and III_k of the reciprocal system λ' and λ units respectively. This is equivalent to the statement that the multiplication table of two reciprocal systems differ only in the fact that the rows and columns of one are interchanged to obtain the other.

As a guide to the construction of the tables of combination for systems in seven units, I will give here the tables for $n = 3, 4, 5, 6$. It should be noted that the order of units in these tables is entirely unimportant. In fact, it often appears that the order of the units which is used in the final multiplication table of the system must be different from the one that appears in the tables of combination in order to put the multiplication table into Scheffer's normal form* for non-quaternion systems. Thus, for example: for $n = 5$, $r = 3$ below, in the second system it is by no means essential that e_1 is in group I with respect to e_4, e_2 in I with respect to e_5, etc., but merely in the system for which $r = 3$ there must be one unit in I and one in II with respect to one of the idempotent units, and one unit in I and one in III with respect to the other idempotent unit.

In the following tables the idempotent units are represented at the top and the non-idempotent units at the left-hand side. When there is no ambiguity, the unit e_k is represented by the subscript k only. The group of e_i with respect to e_k is at the intersection of the k^{th} column and i^{th} row. The space where IV would appear is left blank. As usual, r represents the number of non-idempotent units, and k represents the total number of units in all groups I.

$n = 3$

$r = 1, \ k = 0,$

	2	3
1	II	III

* See my paper, Annalen, loc. cit.

$n=4$

$r=2,\ k=1.$

In this case there is one even unit and the combination given below is the only one that is admissible. The combination

	3	4
1	I	
2	III	II

gives systems reciprocal to the one retained while

	3	4
1		I
2	II	III

	3	4
1		I
2	III	II

are equivalent to the combinations given below and above respectively after an interchange of the units 3 and 4.

The table for $k=0$ is of obvious construction. We have then

	$k=1$		$k=0$			
	3	4	3	4	3	4
1	I		II	III	II	III
2	II	III	II	III	III	II

$n=5$

$r=2$

In this case we can have no even unit as the system would then be reducible. For one skew unit could not connect three idempotent units, and the modulus would fall apart (see 4 p. 224). Thus we have only

$k = 0$

	3	4	5	3	4	5
1	II	III		II	III	
2		II	III		III	II

$r = 3$

	$k = 2$				$k = 1$				$k = 0$			
	4	5	4	5	4	5	4	5	4	5	4	5
1	I		I		I		I		II	III	II	III
2	I			I	II	III	II	III	II	III	II	III
3	II	III	II	III	II	III	III	II	II	III	III	II

$n = 6$

$r = 3$

$k = 1$

	4	5	6	4	5	6	4	5	6	4	5	6
1	I			I				I			I	
2	II	III		II	III		II	III		II	III	
3		II	III		III	II		II	III		III	II

$k = 0$

	4	5	6	4	5	6	4	5	6	4	5	6	4	5	6
1	II	III		II	III			II	III	II		III	III		II
2	II	III		II	III		II	III		II	III		II	III	
3		II	III		III	II		III	II		II	III		II	III

$r = 4$

	$k = 3$				$k = 2$								$k = 1$				$k = 0$					
	5	6	5	6	5	6	5	6	5	6	5	6	5	6	5	6	5	6	5	6	5	6
1	I		I		I		I		I		I		I		I		II	III	II	III	II	III
2	I		I		I		I			I		I	II	III	II	III	II	III	II	III	II	III
3	I			I	II	III	II	III	II	III	II	III	II	III	II	III	II	III	II	III	III	II
4	II	III	II	III	II	III	III	II	II	III	III	II	II	III	III	II	II	III	III	II	III	II

In the combinations given above, it is to be noted that in setting up combinations for the case where

$$\text{(A)} \qquad n = n_1; \quad r = r_1; \quad k = k_1 \neq 0,$$

we may make use of the combinations for the case where

$$\text{(B)} \qquad n = n_1 - k_1; \quad r = r_1 - k_1; \quad k = 0.$$

For example, the combinations for $n = 6$, $r = 4$, $k = 1$ follow immediately from those for $n = 5$, $r = 3$, $k = 0$. In fact, all possible distinct-combinations of the $r_1 - k_1$ skew units which are called for in case (A) are already obtained in case (B). Thus, to get all possible distinct combinations for case (A) we only need to arrange the k_1 even units so as to give the various distinct combinations. This is illustrated in the tables given above in the construction of the case $n = 6$, $r = 4$, $k = 2$ from $n = 4$, $r = 2$, $k = 0$. This principle is also applicable when $k = 0$. For instance, if we wish to write the combinations for

$$n = n_1; \quad r = r_1; \quad k = 0,$$

we may make use of the table for the case

$$n = n_1 - 1; \quad r = r_1 - 1; \quad k = 0,$$

which gives all possible combinations of $r_1 - 1$ skew units, and the proper arrangement of the remaining skew unit with respect to these, is all that remains. This is illustrated in the derivation of the combination for $n = 6$, $r = 4$, $k = 0$ from those for $n = 5$, $r = 3$, $k = 0$. By this device it is possible

to write the tables of combination very rapidly by inspection without including any that are superfluous.

§3.

Tables of Combination for $n = 7$.

Since two skew units can connect at most three idempotent units, all systems for which $r < 3$ are reducible. Thus we first consider

$r = 3$.

There are in this case four idempotent units, a number which is not reached in irreducible systems in less than seven units. Thus for this case the tables of combination must be constructed without the assistance of tables already derived. If one of our three non-idempotent units is even, it leaves only two skew units to connect four idempotent units, which is impossible. Thus all the non-idempotent units must be skew, and we obtain

$k = 0$

	1_3				2_3				3_3				4_3				5_3			
	4	5	6	7	4	5	6	7	4	5	6	7	4	5	6	7	4	5	6	7
1	II	III			II	III			II	III			II	III			II	III		
2	II		III		II		III		II		III		II		III		III		II	
3	II			III	III			II		III		II		II		III		II		III

$r = 4$.

Since two skew units are sufficient to connect three idempotent, we may have in this case at most two even units. For the case $k = 2$ we make use of the table for $n = 5$, $r = 2$, $k = 0$.

$k=2$

	1_4			2_4			3_4			4_4			5_4			6_4			7_4			8_4		
	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7
1	I			I				I			I		I			I			I			I		
2	I			I				I			I			I			I				I			I
3	II	III		II	III		II	III		II	III		II	III		II	III		II	III		II	III	
4		II	III		III	II		II	III		III	II		II	III		III	II		II	III		III	II

$k=1$

Here we make use of the case $n=6$, $r=3$, $k=0$.

	9_4			10_4			11_4			12_4			13_4			14_4		
	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7
1	I			I			I			I			I				I	
2	II	III		II	III			II	III	II		III	III		II	II	III	
3	II	III		II	III		II	III		II	III		II	III		II	III	
4		II	III		III	II		III	II		II	III		II	III		II	III

	15_4			16_4			17_4			18_4			19_4			20_4		
	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7
1		I			I			I				I			I			I
2	II	III			II	III	II		III	II	III		II	III			II	III
3	II	III		II	III		II	III		II	III		II	III		II	III	
4		III	II		III	II		II	III		II	III		III	II		III	II

$k = 0$

In this case we also use $n = 6$, $r = 3$, $k = 0$.

	21_4			22_4			23_4			24_4			25_4			26_4		
	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7
1	II	III		II	III		II	III		II	III		II		III	II		III
2	II	III		II	III			II	III	II		III	II	III			II	III
3	II	III		II	III		II	III		II	III		II	III		II	III	
4		II	III		III	II		III	II		II	III		III	II		III.	II

	27_4			28_4			29_4			30_4			31_4			32_4		
	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7	5	6	7
1		II	III		II	III	III	II		III	II		III		II		III	II
2	II	III			II	III	II	III			II	III	II	III		II	III	
3	II	III		II	III		II	III		II	III		II	III		II	III	
4		II	III		III	II		III	II		III	II		II	III		III	II

$r=5$.

By use of combinations for $n=3$, $r=1$, $k=0$ and $n=4$, $r=2$, $k=0$ we obtain combinations for $k=4$ and $k=3$.

$k=4$ $k=3$

	1_5		2_5		3_5		4_5		5_5		6_5		7_5	
	6	7	6	7	6	7	6	7	6	7	6	7	6	7
1	I		I		I		I		I		I		I	
2	I		I		I		I		I		I		I	
3	I		I			I	I		I			I		I
4	I			I		I	II	III	II	III	II	III	II	III
5	II	III	II	III	II	III	II	III	III	II	II	III	III	II

By use of combinations for $n=5$, $r=3$, $k=0$ and $n=6$, $r=4$, $k=0$ we obtain the combinations for $k=2$, $k=1$ and $k=0$.

$k=2$ $k=1$ $k=0$

	8_5		9_5		10_5		11_5		12_5		13_5		14_5		15_5		16_5		17_5	
	6	7	6	7	6	7	6	7	6	7	6	7	6	7	6	7	6	7	6	7
1	I		I		I		I		I		I		I		II	III	II	III	II	III
2	I		I			I		I	II	III	II	III	II	III	II	III	II	III	II	III
3	II	III	II	III	II	III	II	III	II	III	II	III	II	III	II	III	II	III	II	III
4	II	III	II	III	II	III	II	III	II	III	II	III	III	II	II	III	II	III	III	II
5	II	III	III	II	II	III	III	II	II	III	III	II	III	II	II	III	III	II	III	II

§4.

Method of Obtaining Multiplication Tables.

In any non-quaternion system the units can be chosen so that the multiplication table is in Scheffers' normal form. This form is characterized as follows. Let $e_1, e_2, \ldots, e_r$ be the non-idempotent units.

Then
$$e_i e_j = \sum_{k=1}^{l-1} \gamma_{ijk} e_k \qquad i, j \leqq r$$

where l is the lesser of i and j. Also

$$e_i e_{r+s} = e_{r+s} e_i = 0 \qquad i \leqq r;\ 0 < s \leqq n - r$$

except when e_i is in groups I, II or III with respect to e_{r+s} when the value of the product is determined by the group in which e_i is found with respect to e_{r+s}. Also,

$$e_{r+s}\, e_{r+t} = 0, \qquad s \neq t,\ 0 < s,\ t \leqq n - r.$$
$$e_{r+s}\, e_{r+s} = e_{r+s}, \qquad 0 < s \leqq n - r.$$

The units of any system in which the independent idempotent numbers are taken as units, are also subject to multiplicative properties expressed by the following table, which gives the group to which the product (when non-vanishing) of two units of given groups must belong:

(A)

	I	II	III	IV
I	I	II	0	0
II	0	0	I	II
III	III	IV	0	0
IV	0	0	III	IV

Thus, for instance, if e_i and e_j are in groups II and III respectively with respect to a given idempotent unit, their product $e_i e_j$ will be in group I with respect to the same unit, provided it does not vanish.

Thus the units of any non-quaternion system must fall in Scheffers' normal form and obey table (A) simultaneously. Consequently the first step in deriving the multiplication tables from the tables of combination is to apply the associative law to bring the units into Scheffers' normal form. The units are assumed to obey table (A) at the outset. This process is always possible, and very simple to carry out. For further reduction of parameters we make use of the following principles:

Definition. A system is said to be deleted by a given unit, when that unit is erased from every position which it occupies in the multiplication table of the system.

Definition. A number is nilfactorial with respect to a number β if

$$\alpha\beta = \beta\alpha = 0.$$

6. *If a non-quaternion system is deleted by a unit that is nilfactorial with respect to every non-idempotent unit, the deleted system is associative.*

7. *If a system is deleted by one or more units so that there remains only certain idempotent units and one or more unbroken groups with respect to those units, the deleted system is associative.*

8. *If two systems are deleted by the same method* (*i. e. both under* 6) *and the deleted systems are inequivalent, the original systems are inequivalent.*

Principles 7 and 8 show that when in a table of combination, for instance, three units of group I with respect say to e_7 $(n = 7)$ occur, that the various distinct systems in four units, three of which are in group I with respect to the remaining unit, constitute sub-systems of distinct systems in seven units.

The types of distinct systems in less than six units are taken from Scheffers' paper already quoted.

§5.

Enumeration of Inequivalent Systems in Seven Units.

The tables of combinations given in §3 determine the portions of the corresponding multiplication tables which involve the idempotent units in products either with each other or with the non-idempotent units. Thus 1_3 leads to the table following where table (A) has been applied to determine the products

of non-idempotent units, and the system is assumed to be in Scheffers' normal form.

	1	2	3	4	5	6	7
1	0	0	0	0	1	0	0
2	0	0	0	0	0	2	0
3	0	0	0	0	0	0	3
4	1	2	3	4	0	0	0
5	0	0	0	0	5	0	0
6	0	0	0	0	0	6	0
7	0	0	0	0	0	0	7

In the following enumeration the portion of the multiplication table involving the idempotent units is not displayed in matricular form as above, but is to be supplied by the reader from the tables of combination. The non-vanishing products of non-idempotent units are given, which, together with the corresponding table of combination, determine the systems completely. Thus "all vanish" means that all products of non-idempotent units among themselves vanish, while "$e_2e_2 = e_1$" means that all products of non-idempotent units among themselves with this exception vanish.

Thus:

$r = 0$

$k = 0$

1_3. All vanish.

2_3. All vanish.

3_3. All vanish.

4_3. All vanish.

5_3. All vanish.

$r = 4$

$k = 2$. 1_4. Since $\mathrm{I} \cdot \mathrm{II} = \mathrm{II}$ by table (A), and since no product of two non-idempotent units can contain either one of those units, we have $\mathrm{I} \cdot \mathrm{II} = 0$. Similarly $\mathrm{II} \cdot \mathrm{III} = \mathrm{I}$, but as there is no unit in I with respect to e_6, we have

II · III = 0. Thus, by 8 and systems III_1 and III_2 of Scheffers' enumeration, we have

1_4·1. $e_2e_2 = e_1$.
1_4·2. All vanish.
2_4·1. $e_2e_2 = e_1$.
2_4·2. All vanish.
3_4·1. $e_2e_2 = e_1$.
3_4·2. All vanish.
4_4. All vanish.
5_4. All vanish.
6_4. All vanish.
7_4. All vanish.
8_4. All vanish.

$k = 1$. 9_4. In this case, if we take the units in the order given in the table of combination, we should have $e_1e_3 = e_2$, which is not according to Scheffers' normal form. But on interchange of the units e_1 and e_2 we have an equivalent system,

9_4·1. $e_2e_3 = e_1$.
9_4·2. All vanish.
10_4. Interchange e_1 and e_2 and we obtain
10_4·1. $e_2e_3 = e_1$.
10_4·2. All vanish.
11_4. All vanish.
12_4·1. $e_3e_4 = e_2$.
12_4·2: All vanish.
13_4. All vanish.
14_4. Interchange e_1 and e_2 and we obtain
14_4·1. $e_3e_2 = e_1$.
14_4·2. All vanish.
15_4. Interchange e_1 and e_2 and we obtain
15_4·1. $e_3e_2 = e_1$.
15_4.2. All vanish.
16_4·1. $e_2e_4 = e_1$.
16_4·2. All vanish.
17_4·1. $e_3e_4 = e_1$.

$17_4 \cdot 2$. All vanish.

18_4. All vanish.

19_4. All vanish.

$20_4 \cdot 1$. $e_4 e_2 = e_1$.

$20_4 \cdot 2$. All vanish.

$k = 0$, 21_4. All vanish.

22_4. All vanish.

23_4. All vanish.

$24_4 \cdot 1$. $e_3 \cdot e_4 = e_2$.

$24_4 \cdot 2$. All vanish.

25_4. Interchange e_1 and e_2 and we obtain

$25_4 \cdot 1$. $e_2 e_4 = e_1$.

$25_4 \cdot 2$. All vanish.

$26_4 \cdot 1$. $e_3 e_2 = e_1$.

$26_4 \cdot 2$. All vanish.

27_4. All vanish.

28_4. All vanish.

29_4. All vanish.

30_4. All vanish.

31_4. All vanish.

32_4. All vanish.

$r = 5$.

$k = 4$. 1_5. We get as many inequivalent systems as there are inequivalent systems in five units one of which is idempotent.

Thus, $1_5 \cdot 1$. $e_2 e_4 = e_4 e_2 = e_3^2 = e_1$; $e_3 e_4 = e_4 e_3 = e_2$; $e_4^2 = e_3$.

$1_5 \cdot 2$. $e_2 e_3 = e_3 e_2 = e_3 e_4 = - e_4 e_3 = e_4^2 = e_1$; $e_3^2 = e_2$.

$1_5 \cdot 3$. $e_2 e_3 = e_3 e_2 = e_4^2 = e_1$; $e_3^2 = e_2$.

$1_5 \cdot 4$. $e_2 e_3 = e_3 e_2 = e_3 e_4 = - e_4 e_3$; $e_3^2 = e_2$.

$1_5 \cdot 5$. $e_2 e_3 = e_3 e_2 = e_1$; $e_3^2 = e_2$.

$1_5 \cdot 6$. $e_3 e_4 = \dfrac{1}{\lambda} e_4 e_3 = e_1$; $e_4^2 = e_2$.

$1_5 \cdot 7$. $e_3^2 = e_1$; $e_4^2 = e_2$.

$1_5 \cdot 8$. $e_3 e_4 = e_1$; $e_4 e_3 = e_2$.

$1_5 . 9$. $e_3 e_4 = e_1 + e_2$; $e_4 e_3 = - e_1 + e_2$; $e_4^2 = e_1$.

$1_5 \cdot 10$. $e_3^2 = e_1$; $e_3e_4 = -e_4e_3 = e_2$; $e_4^2 = e_2 + \lambda e_1$.

$1_5 \cdot 11$. $e_2^2 = e_3^2 = e_4^2 = e_1$.

$1_5 . 12$. $e_3^2 = e_4^2 = e_1$.

$1_5 \cdot 13$. $e_4^2 = e_1$.

$1_5 \cdot 14$. $e_2^2 = e_3^2 = e_4^2 = \frac{1}{\lambda} e_3e_4 = -\frac{1}{\lambda} e_4e_3 = e_1$.

$1_5 \cdot 15$. $e_2^2 = e_3e_4 = -e_4e_3 = e_4^2 = e_1$.

$1_5 \cdot 16$. $e_2^2 = e_3e_4 = -e_4e_3 = e_1$,

$1_5 \cdot 17$. $e_3^2 = e_4^2 = \frac{1}{\lambda} e_3e_4 = -\frac{1}{\lambda} e_4e_3 = e_1$.

$1_5 \cdot 18$. $e_3e_4 = -e_4e_3 = e_4^2 = e_1$.

$1_5 \cdot 19$. $e_2e_3 = e_3e_2 = e_4e_3 = -e_4e_3 = e_4^2 = e_1$.

$1_5 \cdot 20$. $e_2e_3 = e_3e_2 = e_4e_3 = -e_4e_3 = e_1$.

$1_5 \cdot 21$. $e_3e_4 = -e_4e_3 = e_1$.

$1_5 \cdot 22$. All vanish.

2_5. We make similar use of Scheffers' list of systems in four units and obtain,

$2_5 \cdot 1$. $e_2e_3 = e_3e_2 = e_1$; $e_3^2 = e_2$.

$2_5 \cdot 2$. $e_2^2 = e_2e_3 = -e_3e_2 = \frac{1}{\lambda} e_3^2 = e_1$.

$2_5 \cdot 3$. $e_2^2 = e_3^2 = e_1$.

$2_5 \cdot 4$. $e_3^2 = e_1$.

$2_5 \cdot 5$. $e_2e_3 = -e_3e_2 = e_1$.

$2_5 \cdot 6$. All vanish.

$3_5 \cdot 1$. $e_2^2 = e_1$; $e_4^2 = e_3$.

$3_5 \cdot 2$. $e_2^2 = e_1$.

$3_5 \cdot 3$. All vanish.

$k = 3$ $4_5 \cdot 1$. We get by use of the associative law on the products $e_2e_3e_5$ and $e_3^2e_5$, after interchanging e_3 and e_4.

$4_5 \cdot 1$. $e_2e_4 = e_4e_2 = e_1$; $e_4^2 = e_2$; $e_4e_5 = e_3$.

$4_5 \cdot 2$. $e_2e_4 = e_4e_2 = e_1$; $e_4^2 = e_2$.

4_5·3–14. Interchange e_4 and e_2 and obtain

4_5·3. $e_4^2 = e_4e_3 = -e_3e_4 = \dfrac{1}{\lambda} e_3^2 = e_1$; $e_4e_5 = e_2$; $e_3e_5 = \lambda_1 e_2$.

4_5·4. $e_4^2 = e_4e_3 = -e_3e_4 = \dfrac{1}{\lambda} e_3^2 = e_1$; $e_3e_5 = e_2$.

4_5·5. $e_4^2 = e_4e_3 = -e_3e_4 = \dfrac{1}{\lambda} e_3^2 = e_1$.

4_5·6. $e_4^2 = e_3^2 = e_1$; $e_4e_5 = \dfrac{1}{\lambda} e_3e_5 = e_2$.

4_5·7. $e_4^2 = e_3^2 = e_1$; $e_4e_5 = e_2$.

4_5·8. $e_4^2 = e_3^2 = e_1$.

4_5·9. $e_3^2 = e_1$; $e_4e_5 = e_2$.

4_5·10. $e_3^2 = e_1$.

4_5·11. $e_4e_3 = -e_3e_4 = e_1$; $e_3e_5 = e_2$.

4_5·12. $e_4e_3 = -e_3e_4 = e_1$.

4_5·13. $e_4e_5 = e_2$.

4_5·14. All vanish.

5_5. No interchange is required.

5_5·1. $e_2e_3 = e_3e_2 = e_1$; $e_3^2 = e_2$; $e_4e_5 = e_1$.

5_5·2. $e_2e_3 = e_3e_2 = e_1$; $e_3^2 = e_2$.

5_5·3. $e_2e_3 = -e_3e_2 = e_2^2 = \dfrac{1}{\lambda} e_3^2 = e_1$; $e_4e_5 = e_1$.

5_5·4. $e_2e_3 = -e_3e_2 = e_2^2 = \dfrac{1}{\lambda} e_3^2 = e_1$.

5_5·5. $e_2^2 = e_3^2 = e_4e_5 = e_1$.

5_5·6. $e_2^2 = e_2^2 = e_1$.

5_5·7. $e_3^2 = e_1$; $e_4e_5 = e_2$.

5_5·8. $e_3^2 = e_1$; $e_4e_5 = e_1$.

5_5·9. $e_3^2 = e_1$.

5_5·10. $e_2e_3 = -e_3e_2 = e_4e_5 = e_1$.

5_5·11. $e_2e_3 = -e_3e_2 = e_1$.

5_5·12. $e_4e_5 = e_1$.

5_5·13. All vanish.

$6_5{\cdot}1$–6. Interchange e_2 and e_4.
$6_5{\cdot}1$. $e_4^2 = e_1$; $e_4e_5 = e_2$; $e_5e_3 = e_2$.
$6_5{\cdot}2$. $e_4^2 = e_1$; $e_4e_5 = e_2$.
$6_5{\cdot}3$. $e_4^2 = e_1$; $e_5e_3 = e_2$.
$6_5{\cdot}4$ $e_4^2 = e_1$.
$6_5{\cdot}5$. $e_4e_5 = e_2$; $e_5e_3 = e_2$.
$6_5{\cdot}6$. $e_4e_5 = e_2$.
$6_5{\cdot}7$. All vanish.

$7_5{\cdot}1$. $e_2^2 = e_1$; $e_5e_4 = e_3$; $e_4e_5 = e_1$.
$7_5{\cdot}2$. $e_2^2 = e_1$; $e_5e_4 = e_3$.
$7_5{\cdot}3$. $e_2^2 = e_1$; $e_4e_5 = e_1$.
$7_5{\cdot}4$. $e_2^2 = e_1$.
$7_5{\cdot}5$. $e_4e_5 = e_1$; $e_5e_4 = e_3$.
$7_5{\cdot}6$. $e_4e_5 = e_1$.
$7_5{\cdot}7$. All vanish.

$k = 2$ $8_5{\cdot}1$–9. Interchange cyclically $e_1e_4e_2e_5e_3$.
$8_5{\cdot}1$. $e_5^2 = e_4$; $e_4e_3 = e_5e_2 = e_1$; $e_5e_3 = e_2$.
$8_5{\cdot}2$. $e_5^2 = e_4$; $e_5e_3 = e_2$.
$8_5{\cdot}3$. $e_5^2 = e_4$; $e_5e_2 = e_1$.
$8_5{\cdot}4$. $e_5^2 = e_4$.
$8_5{\cdot}5$. $e_4e_3 = e_1$; $e_5e_3 = e_2$.
$8_5{\cdot}6$. $e_4e_3 = e_1$; $e_5e_2 = e_1$.
$8_5{\cdot}7$. $e_4e_3 = e_1$.
$8_5{\cdot}8$. All vanish.

9_5.1–8. Interchange e_2 and e_3.
$9_5{\cdot}1$. $e_3^2 = e_1$; $e_4e_5 = e_1$.
$9_5{\cdot}2$. $e_3^2 = e_1$.
$9_5{\cdot}3$. $e_3^2 = e_1$; $e_3e_4 = e_2$; $e_4e_5 = e_1$.
$9_5{\cdot}4$. $e_3^2 = e_1$; $e_3e_4 = e_2$.
$9_5{\cdot}5$. $e_3e_4 = e_2$; $e_4e_5 = e_1$.
$9_5{\cdot}6$. $e_3e_4 = e_2$.

$9_5{\cdot}7$. $e_3e_5 = e_1$; $e_4e_5 = e_3$.
$9_5{\cdot}8$. $e_3e_5 = e_1$.
$9_5{\cdot}9$. All vanish.

$10_5{\cdot}1$–6. Interchange e_1 with e_3, and e_2 with e_4.
$10_5{\cdot}1$. $e_3e_5 = e_2$; $e_5e_4 = e_1$.
$10_5{\cdot}2$. $e_3e_5 = e_2$; $e_5e_4 = e_2$.
$10_5{\cdot}3$. $e_3e_5 = e_2$.
$10_5{\cdot}4$. $e_3e_2 = e_1$; $e_5e_4 = e_1$.
$10_5{\cdot}5$. All vanish.

$11_5{\cdot}1$–3. Interchange e_1 and e_3.
$11_5{\cdot}1$. $e_3e_4 = e_1$; $e_4e_2 = e_1$.
$11_5{\cdot}2$. $e_3e_4 = e_1$; $e_5e_4 = e_2$.
$11_5{\cdot}3$. $e_3e_4 = e_1$.
$11_5{\cdot}4$. $e_4e_5 = e_1$; $e_5e_3 = e_2$.
$11_5{\cdot}5$. $e_4e_5 = e_1$.
$11_5{\cdot}6$. $e_4e_5 = e_1$; $e_5e_4 = e_2$.
$11_5{\cdot}7$. All vanish.

$k = 1$ $12_5{\cdot}1$–3. Interchange cyclically $e_1e_4e_3e_2$.
$12_5{\cdot}1$. $e_4e_2 = e_1$; $e_4e_5 = e_3$.
$12_5{\cdot}2$. $e_4e_5 = e_3$.
$12_5{\cdot}3$. All vanish.

$13_5{\cdot}1$–3. Interchange e_1 and e_2.
$13_5{\cdot}1$. $e_2e_3 = e_1$.
$13_5{\cdot}2$. $e_2e_4 = e_1$; $e_3e_5 = e_2$.
$13_5{\cdot}3$. $e_4e_5 = e_2$.
$13_5{\cdot}4$. All vanish.

$14_5{\cdot}1$. Interchange e_1 and e_2.
$14_5{\cdot}1$. $e_1e_3 = e_2$.
$14_5{\cdot}2$. Interchange cyclically $e_1e_2e_4$.
$14_5{\cdot}2$. $e_4e_5 = e_1$; $e_5e_2 = e_1$.

$14_5 \cdot 3$. $e_2e_5 = e_1$; $e_3e_4 = e_1$.

$14_5 \cdot 4$. $e_2e_5 = e_1$.

$14_5 \cdot 5$. All vanish.

$k = 0$ 15_5. All vanish.

16_5. All vanish.

17_5. All vanish.

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